Published by Institute of Physics Publishing for SISSA

Received: *July 17, 2007*
Accepted: *October 26, 2007*
Published: *October 30, 2007*

AdS_3 partition functions reconstructed

Jan Manschot

Institute for Theoretical Physics, University of Amsterdam,
Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands
E-mail: manschot@science.uva.nl

Abstract: For pure gravity in AdS_3, Witten has given a recipe for the construction of holomorphically factorizable partition functions of pure gravity theories with central charge $c = 24k$. The partition function was found to be a polynomial in the modular invariant j-function. We show that the partition function can be obtained instead as a modular sum which has a more physical interpretation as a sum over geometries. We express both the j-function and its derivative in terms of such a sum.

Keywords: AdS-CFT Correspondence, Black Holes.

JHEP10(2007)103

 http://jhep.sissa.it/archive/papers/jhep102007103/jhep102007103.pdf

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1. Introduction

Partition functions of gravity in three dimensions with a negative cosmological constant are strongly constrained by modular invariance. Witten recently used this constraint in ref. [1] to construct partition functions for pure gravity theories which allow a holomorphically factorized partition function. The constraint of modular invariance is strong enough to determine a conformal field theory partition function completely from the lowest terms of its Laurent expansion, as demonstrated earlier in [2]. Ref. [1] uses these lowest terms to express the partition function as a polynomial in the modular invariant $j(\tau)$. However, the partition function as a polynomial in $j(\tau)$ does not display any apparent connection to the gravity path integral.

We would like to emphasize in this note that each of the partition functions obtained in ref. [1] can be written as a sum over a coset of the modular group. This sum has a clear interpretation as a sum over geometries. Ref. [2] first wrote the partition function in this way for the D1-D5 system, where the expansion was given the name "Farey tail." In the past year there have been several applications of these techniques to $\mathcal{N} = 2$ supersymmetric black holes in four dimensions [3 − 6].

2. Gravity action

Ref. [1] is mainly concerned with pure gravity without a gravitational Chern-Simons term. This gives rise to a partition function with a holomorphic as well as an anti-holomorphic dependence. A subclass of partition functions are those which can be holomorphically factorized. The holomorphic or anti-holomorphic part can be studied independently in those situations.

In this note, we choose to restrict to theories whose partition functions are holomorphic by adding an appropriate Chern-Simons term to the standard Einstein-Hilbert action. The action is the common Einstein-Hilbert action plus the gravitational Chern-Simons term (in Euclidean signature) [4]

$$S_{\text{grav}} = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R-\frac{2}{l^2}\right)+\frac{k'}{4\pi}\int d^3x\Omega_3(\omega), \tag{2.1}$$

where $\Omega_3(\omega)$ is the holomorphic Chern-Simons form,

$$\Omega_3(\omega) = \omega\wedge d\omega + \omega\wedge\omega\wedge\omega. \tag{2.2}$$

We introduce a gauge field $A_L = \omega -^* e/l$ and $A_R = \omega +^* e/l$. The action in terms of these variables is

$$S_{\text{grav}} = \frac{k_L}{4\pi}\int A_L\wedge dA_L + \frac{2}{3}A_L\wedge A_L\wedge A_L - \frac{k_R}{4\pi}\int A_R\wedge dA_R + \frac{2}{3}A_R\wedge A_R\wedge A_R, \tag{2.3}$$

with $k_L = \frac{l}{16G}+\frac{k'}{2}$ and $k_R = \frac{l}{16G}-\frac{k'}{2}$. Our aim is to study a holomorphic theory, so we take $k_R = 0$, which gives $k_L = \frac{l}{8G} = k$. Quantum mechanical consistency requires k to be an integer.

Gravity in three dimensions has no local degrees of freedom. Different geometries are determined by globally different identifications. The path integral therefore reduces to a sum over these identifications. We can determine the action for different geometries. The action of thermal AdS_3 with k_R equal to 0 is [7]

$$S = 2\pi i k\tau. \tag{2.4}$$

The action of the BTZ black hole is

$$S = -\frac{2\pi i k}{\tau}. \tag{2.5}$$

The action of the BTZ black hole and thermal AdS_3 are related by the transformation $\tau \to -\frac{1}{\tau}$, which is a generator of $\text{SL}(2,\mathbb{Z})$. Ref. [2] shows that the geometries of AdS_3 are in one-to-one correspondence with the coset $\Gamma_\infty\backslash\,\text{SL}(2,\mathbb{Z})$, where Γ_∞ is the group of "translations" given by $\begin{pmatrix}1 & r\\ 0 & 1\end{pmatrix}$. Any element $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_\infty\backslash\,\text{SL}(2,\mathbb{Z})$ is determined by a choice of two relatively prime integers c and d. This set of different geometries comes about as different choices of the primitive contractible cycle when Euclidean AdS_3 is viewed as a filled torus [2]. The action of these other geometries is given by $2\pi i\left(\frac{a\tau+b}{c\tau+d}\right)$. The gravity partition function is now given by

$$Z_k(\tau) = \sum_{\text{geometries}} e^{-S} = \sum_{\Gamma_\infty\backslash\,\text{SL}(2,\mathbb{Z})} e^{-2\pi i k\frac{a\tau+b}{c\tau+d}}M(c\tau+d), \tag{2.6}$$

where $M(c\tau+d)$ is some measure factor. Such sums over the modular group are known as Poincaré series in the mathematical literature. Modding out the translations Γ_∞ from $\text{SL}(2,\mathbb{Z})$ is necessary for convergence in such sums.

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3. Conformal field theory partition function

The AdS/CFT correspondence relates the degrees of freedom in the bulk of AdS space including gravity to a conformal field theory on the boundary. Ref. [1] argues that for pure gravity the whole partition function can be constructed from knowledge of the ground state. The ground state energy is given by $-k = -c/24$, where c is the left moving central charge. States other than the vacuum and its descendants must be related to black holes, because gravity in three dimensions has no local degrees of freedom. Primary states other than the vacuum do not have negative energy because black holes with negative mass do not exist. Therefore, all polar terms (i.e. q^{-n}, $n > 0$) in the partition function are the vacuum and its descendants. The vacuum $|0\rangle$ is primary and $\mathrm{SL}(2,\mathbb{R})$ invariant and is thus annihilated by L_n for $n \geq -1$. Acting with creation operators L_{-n}, $n \geq 2$ generates a tower of states with partition function

$$Z_{\mathrm{subset},k}(\tau) = q^{-k} \prod_{n=2}^{\infty} \frac{1}{(1-q^n)}, \tag{3.1}$$

We gave this partition function the subscript "subset" because it represents only a subset of the total number of states in the theory. A direct way to see this is that the partition function is not modular invariant. Ref. [1] constructs a modular invariant partition function with the required polar behavior with the use of $J(\tau) = j(\tau) - 744$, the unique modular invariant with a polar term q^{-1} and vanishing q^0 term. $j(\tau)$ is given by

$$j(\tau) = \frac{1728 E_4(\tau)^3}{\Delta(\tau)} = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n) q^n, \tag{3.2}$$

where $\Delta = \eta(\tau)^{24}$, and $E_4(\tau)$ is the familiar Eisenstein series of weight 4. The partition function for $k = 1$ is equal to $Z_1(\tau) = J(\tau)$. The partition functions for larger values of k become polynomials in $J(\tau)$.

The exact Fourier coefficients of $j(\tau)$ can be determined with the circle method introduced by Rademacher.[1] This method to determine Fourier coefficients was originally obtained for modular forms of negative weight and with a polar part. It also turned out to be very useful for the determination of the Fourier coefficients of $j(\tau)$ which has weight 0. The coefficients $c(n)$ are given by the infinite sum [9, 10]

$$c(n) = \frac{2\pi}{\sqrt{n}} \sum_{m=1}^{\infty} \frac{K_m(n)}{m} I_1\left(\frac{4\pi\sqrt{n}}{m}\right), \tag{3.3}$$

where $K_m(n)$ is the Kloosterman sum

$$K_m(n) = \sum_{d \in (\mathbb{Z}/m\mathbb{Z})^*} \exp\left(\frac{2\pi i (nd + \bar{d})}{m}\right), \qquad d\bar{d} = -1 \bmod m, \tag{3.4}$$

and $I_\nu(z)$ is the Bessel function defined by

$$I_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-\nu-1} e^{t + \frac{z^2}{4t}} dt, \qquad (c > 0, \mathrm{Re}(\nu) > 0). \tag{3.5}$$

[1]Ref. [8] contains a clear exposition of the circle method applied to the Dedekind η-function.

4. Partition function as a sum over geometries

We would like to relate the conformal field theory partition function to the gravity partition function (2.6) in a spirit similar to [2]. There exist in fact two sums over integers c and d that are relatively prime ($(c,d)=1$), which are related to the modular invariant $J(\tau)$. We will comment on both.

Refs. [12, 11] give for $J(\tau)$

$$J(\tau) = -12 + \lim_{K\to\infty} \frac{1}{2} \sum_{|c|\le K} \sum_{\substack{|d|\le K \\ (c,d)=1}} \exp 2\pi i \left(-\frac{a\tau+b}{c\tau+d}\right) - s(c,d), \qquad ad-bc=1, \tag{4.1}$$

where $s(c,d)$ is defined as $\exp 2\pi i \left(-\frac{a}{c}\right)$ when $c\neq 0$ and otherwise 0. The subtraction of $s(c,d)$ is necessary for convergence. The order of summation over c and d is important in this case. For every integer pair (c,d), a and b are chosen to satisfy $ad-bc=1$. Two differences with the gravity partition function, eq. (2.6), are the subtraction of $s(c,d)$ and the dependence on the order of summation of c and d. The sum over (c,d) can still be interpreted as different choices of the primitive contractible cycle as in eq. (2.6), but the subtraction of $s(c,d)$ might be harder to interpret from the gravity point of view. The q^0 term is not determined by the modular sum, because it is itself a modular form of weight 0. Similar sums are known for other modular forms with negative (integer) weight, although they might transform with a shift [11].

A way to cure the discrepancies between the gravity partition function and the sum in (4.1) is to consider the so-called Farey transform of the partition function. The Farey transform of the weight zero partition function $Z_k(\tau)$ is simply the derivative $DZ_k(\tau)$, where we defined the differential operator

$$D = \frac{1}{2\pi i}\frac{d}{d\tau}.$$

Thus the Farey transform of $J(\tau)$ is $DJ(\tau)$. The inverse transform gives back $J(\tau)$ up to the constant term. Calculation of the Fourier coefficients of the relevant Poincaré series [13] shows that these are equal to those of the Farey transformed partition function. Ref. [9] gives the Poincaré series for $DJ(\tau)$ as

$$DJ(\tau) = -\frac{1}{2} \sum_{\Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \frac{\exp 2\pi i \left(-\frac{a\tau+b}{c\tau+d}\right)}{(c\tau+d)^2}. \tag{4.2}$$

The fact that $DJ(\tau)$ is a weight 2 modular form makes a convergent series possible irrespective of the order of the summation over c and d. The sum is now much more reminiscent of eq. (2.6), giving it a natural interpretation as a sum over geometries. The measure factor introduced in eq. (2.6) is determined to be $M(c\tau+d) = -\frac{1}{2}(c\tau+d)^{-2}$. In this sense the sum gives a physical explanation of the modular invariance and shows moreover how the complete partition function is obtained from knowledge of the polar part of the partition function.

The construction of the transformed partition function for larger values of k is straightforward now that we know it for $k = 1$. Take the derivative of $Z_{\text{subset},k}(\tau)$ and perform a Laurent expansion up to the constant term (similarly to [1]):

$$D\tilde{Z}_k(\tau) = \sum_{-k \leq r < 0} a(r) q^r. \tag{4.3}$$

Then the derivative of the total partition function is given by

$$DZ(\tau) = -\frac{1}{2} \sum_{-k \leq r < 0} \sum_{(c,d)=1} a(r) \frac{\exp 2\pi i \left(r \frac{a\tau + b}{c\tau + d} \right)}{(c\tau + d)^2}. \tag{4.4}$$

Partition functions for larger values of k can also be written as sums analogous to eq. (4.1). The resulting series have with eq. (4.4) in common, that they are both modular sums of polar terms. The states corresponding to the polar terms have a physical interpretation as states which are not sufficiently massive to form black holes. The mass of a black hole is given in the holomorphic case by $M = \frac{1}{l}(L_0 - \frac{c_L}{24})$ and a black hole is only formed when $M \geq 0$. Note that in principle, terms q^r $(r > 0)$ could be included in the sum. The sum over these terms would vanish since cusp forms do not exist for weight 0 and 2.

We have given arguments to interpret holomorphic partition functions as sums over geometries. However, ref. [1] does not consider holomorphic partition functions but holomorphic factorizable partition functions. An example of such a partition function is $Z_1(\tau, \bar{\tau}) = |J(\tau)|^2$. Application of the sums in eqs. (4.1) or (4.2) leads to a sum over (c, d) and $(\tilde{c}, \tilde{d})$, one pair for the holomorphic side and one for the anti-holomorphic side.[2] Only the terms with $(c, d) = (\tilde{c}, \tilde{d})$ correspond to classical geometries. This raises the puzzle that holomorphically factorizable partition functions require states which are difficult to interpret classically.

5. Conclusion

We have considered the question of how to construct holomorphic partition functions of pure gravity in AdS_3 for given central charge. We emphasized the fact that the partition function can be written as a sum over $\Gamma_\infty \backslash \text{SL}(2, \mathbb{Z})$. We presented two such sums: one for $J(\tau)$, and one for its Farey transform $DJ(\tau)$. These sums are easily extended to partition functions for larger values of the central charge. In this way, the partition functions display a closer relation with the gravity path integral.

The appearance of the Farey transformed partition function and why it is more reminiscent of the gravity path integral remains mysterious (see also [2, 6]). A second puzzle is the contribution of geometries without a proper classical realization to holomorphically factorizable partition functions.

Acknowledgments

I would like to thank Erik Verlinde for encouragement and stimulating discussions. This research is supported by the Foundation of Fundamental Research on Matter (FOM).

[2] I would like to thank E. Witten and the referee for bringing this point to my attention.

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